A new formula to compute the n’th binary digit of pi

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We describe here a mean to find formulas similar to those in [1]. We show in particular that

\[
\pi = \frac{1}{2^6} \sum_{n=0}^{\infty} \frac{(-1)^n}{210n} \left( - \frac{2^5}{4n+1} + \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{1}{10n+9} \right). \tag{1}
\]

This formula is very interesting because, with the algorithm described in [1], it enables us to compute the n’th binary digit of \(\pi\) 43% faster than the previous known formula [1]:

\[
\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left( \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \tag{2}
\]

The method to get formulas such as (1) is in fact very simple. We use that

\[ -\ln(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } |x| < 1 \]

and

\[ \text{atan}(y) = \text{Im}[-\ln(1 - i.y)] \]

for \(y\) real.

In particular

\[ \text{atan}\left(\frac{1}{a-1}\right) = \text{Im}[\ln\left(1 - \frac{(1+i)}{a}\right)] \tag{3} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n2^{2n}}{a^{4n+3}} \left( \frac{a^2}{4n+1} + \frac{2a}{4n+2} + \frac{2}{4n+3} \right) \]

and

\[ \text{atan}\left(\frac{1}{a+1}\right) = \text{Im}[\ln\left(1 - \frac{(i-1)}{a}\right)] \tag{4} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n2^{2n}}{a^{4n+3}} \left( \frac{a^2}{4n+1} - \frac{2a}{4n+2} + \frac{2}{4n+3} \right). \]
With \( a \leftarrow 2 \) in (3) we get
\[
\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \left( \frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3} \right)
\]
(5)
which is mentioned in [1].

Some classical arctangent relations give interesting results:
\[
\frac{\pi}{4} = 2 \atan\left( \frac{1}{2} \right) - \atan\left( \frac{1}{7} \right)
\]
(6)
\[
\frac{\pi}{4} = 2 \atan\left( \frac{1}{3} \right) + \atan\left( \frac{1}{7} \right)
\]
(7)
\[
\frac{\pi}{4} = 2 \atan\left( \frac{1}{2} \right) - \atan\left( \frac{1}{9} \right) - \atan\left( \frac{1}{32} \right)
\]
(8)
\[
\frac{\pi}{4} = \atan\left( \frac{1}{2} \right) + \atan\left( \frac{1}{3} \right)
\]
(9)

In particular, we obtain from (6) and (3)
\[
\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^n} - \frac{1}{64} \sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n} \left( \frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3} \right)
\]
which gives (1) by reordering the terms.

The existence of a formula faster than (1) to calculate the \( n \)th binary digit of \( \pi \) remains an open question.

References