A new formula to compute the n'th binary digit of pi

Fabrice Bellard

January 20, 1997

We describe here a mean to find formulas similar to those in [1]. We show in particular that

$$\begin{aligned}\pi &= \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}- \\ &\quad \frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}) \quad . \end{aligned} \tag{1}$$

This formula is very interesting because, with the algorithm described in [1], it enables us to compute the nth binary digit of π 43% faster than the previous known formula [1]:

$$\pi = \sum_{n=0}^{\infty}\frac{1}{16^n}(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}) \tag{2}$$

The method to get formulas such as (1) is in fact very simple. We use that

$$-ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} \quad \text{for } |x|<1$$

and

$$atan(y) = Im[-ln(1-i.y)]$$

for y real.

In particular

$$\begin{aligned} atan(\frac{1}{a-1}) &= Im[-ln(1-\frac{(1+i)}{a})] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{a^{4n+3}}(\frac{a^2}{4n+1}+\frac{2a}{4n+2}+\frac{2}{4n+3}) \end{aligned} \tag{3}$$

and

$$\begin{aligned} atan(\frac{1}{a+1}) &= Im[-ln(1-\frac{(i-1)}{a})] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}}{a^{4n+3}}(\frac{a^2}{4n+1}-\frac{2a}{4n+2}+\frac{2}{4n+3}) \quad . \end{aligned} \tag{4}$$

With $a \leftarrow 2$ in (3) we get

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}) \tag{5}$$

which is mentionned in [1].

Some classical arctangent relations give interesting results:

$$\begin{aligned} \frac{\pi}{4} &= 2atan(\frac{1}{2}) - atan(\frac{1}{7}) & (6) \\ \frac{\pi}{4} &= 2atan(\frac{1}{3}) + atan(\frac{1}{7}) & (7) \\ \frac{\pi}{4} &= 2atan(\frac{1}{2}) - atan(\frac{1}{9}) - atan(\frac{1}{32}) & (8) \\ \frac{\pi}{4} &= atan(\frac{1}{2}) + atan(\frac{1}{3}) \quad . & (9) \end{aligned}$$

In particular, we obtain from (6) and (3)

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^n} - \frac{1}{64}\sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n}(\frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3}) \quad ,$$

which gives (1) by reordering the terms.

The existence of a formula faster than (1) to calculate the nth binary digit of π remains an open question.

References

[1] David H. Bailey, Peter B. Borwein and Simon Plouffe, *On the Rapid Computation of Various Polylogarithmic Constants*, to appear in April 1997 in Mathematics of Computation.